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CONJUGATE FLOWS AND SMOOTH BORES IN A WEAKLY STRATIFIED FLUID

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The problem of steady-state flows in a layer of a continuously stratified fluid is considered. The sufficient condition of existence of families of shear flows that are consistent with the meaning of the laws of conservation of mass, momentum, and energy with a uniform flow is given. Approximate solutions of the smooth-bore type, which describe the wave transitions for pairs of conjugate flows of the first spectral mode, are obtained.

Smooth internal bores are the stationary wave configurations in the layer of a fluid in the form of continuous transitions between two different horizontal flows on the left and on the right at infinity. The smooth bore in a two-layer fluid "under a lid" is described by the model of the second-approximation shallow-wave theory (Ovsyannikov's model [1]); the weakly nonlinear KdV-asymptotic solution was obtained by Funakoshi [2], and the existence of the corresponding exact solutions of the Euler equations was shown in [3–5]. In the case of continuous stratification, bore-type approximate solutions were obtained in [6, 7]. In laboratory experiments, the bore was observed for the case of a two-layer [8] and continuous [9] density distribution. In the present work, the sufficient condition of existence of families of shear flows conjugate to a uniform flow is given for a stratification close to a linear or exponential stratification. The bore structure is investigated for flows which correspond to the first spectral mode of velocities.

1. Initial Equations. We consider the steady-state motion of an inviscid incompressible fluid in the layer "under a lid" $-\infty < x < +\infty$, $0 < y < h$, the scheme of which is shown in Fig. 1. The flow pattern is completely determined by the stream function ψ of the velocity fields $\boldsymbol{u} = (\psi_y, -\psi_x)$: owing to the incompressibility condition, the density ρ is constant along each of the streamlines $\psi(x, y) = \text{const}$, so that $\rho = \rho(\psi)$, and the fluid pressure is determined using the known ρ and ψ from the Bernoulli integral. In this situation, the system of Euler equations is reduced to the Dubreil-Jacotin–Long equation for ψ [10]

$$\rho(\psi)\,\Delta\psi + \rho'(\psi)\left(gy + \frac{1}{2}|\nabla\psi|^2\right) = B'(\psi)$$

with the no-flow conditions at the bottom $y = 0$ and on the lid $y = h$ and the condition $\psi \to \psi^{\pm}(y)$ as $x \to \pm\infty$, where $\psi^+ \neq \psi^-$. It is assumed that, as $x \to -\infty$, the flow tends to a uniform flow with $\boldsymbol{u} = (c, 0)$ and a specified density distribution $\rho_\infty(y)$. In the absence of closed streamlines, we obtain the following form of the function $\rho(\psi)$ and of the Bernoulli function $B(\psi)$:

$$\rho(\psi) = \rho_\infty(\psi/c), \qquad B'(\psi) = \rho'(\psi)\left(\frac{g\psi}{c} + \frac{1}{2}c^2\right).$$

Let N_0 be the characteristic magnitude of the Brunt–Väisälä frequency N, $N^2(y) = -g\,\rho'_\infty(y)/(\rho_\infty(y))$. The basic dimensionless constants in the problem are the Boussinesq parameter σ and the reduced Froude number λ:

$$\sigma = \frac{N_0^2 h}{\pi g}, \qquad \lambda = \frac{\sigma g h}{\pi c^2}.$$

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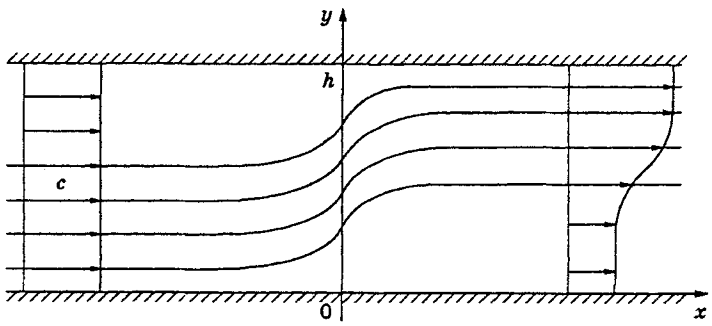

Fig. 1

We introduce the dimensionless variables by using the quantities $h/\pi\sqrt{\sigma}$, h/π, ch/π, and $\rho_\infty(0)$ as the scales for x, y, ψ, and ρ, respectively. We consider the states which have, for $x = -\infty$, the following density distribution in depth:

$$\rho(y,\sigma) = 1 - \sigma y - \sigma^2 \rho_1(y,\sigma). \tag{1.1}$$

Here $\rho_1(0,\sigma) = 0$ according to the choice of the scale for ρ. As particular cases, this dependence includes the linear law and the exponential stratification $\rho = \exp(-\sigma y)$, thus perturbing them by quantities of the order $O(\sigma^2)$ in the weak-stratification limit $\sigma \to 0$. In what follows, it is assumed that the function ρ is defined for $\sigma \in [0, \sigma_0]$ with a certain $\sigma_0 > 0$ and has the following properties.

Condition 1. For $k \geqslant 4$, the function $\rho_1 \in C^k([0,\pi] \times [0,\sigma_0])$ is such that $\rho > 0$ and $\rho_y < 0$ when $(y,\sigma) \in [0,\pi] \times (0,\sigma_0]$.

The problem of bore-type flows is posed as follows. For a specified value of σ, it is required to determine the real positive parameter λ and the function $v(x,y) = \psi(x,y) - y$, which should satisfy the following equations in the band $\Omega = \mathbb{R} \times (0,\pi)$ and at its boundary:

$$F(v, Dv, D^2v, y; \sigma, \lambda) \equiv \operatorname{div}_\sigma (\rho \nabla_\sigma v) - \rho' \left(\sigma^{-1}\lambda v + \frac{1}{2}|\nabla_\sigma v|^2\right) = 0, \quad (x,y) \in \Omega; \tag{1.2}$$

$$v = 0 \qquad (y = 0, \quad y = \pi); \tag{1.3}$$

$$v \to v^\pm, \quad \nabla v \to \nabla v^\pm \qquad (x \to \pm\infty). \tag{1.4}$$

Here $\operatorname{div}_\sigma = \nabla_\sigma \cdot$, $\nabla_\sigma = (\sqrt{\sigma} D_x, D_y)$, $\rho = \rho(y+v,\sigma)$, and $\rho' = \rho_y(y+v,\sigma)$. The expression $v^-(y) \equiv 0$ corresponds to a uniform flow in (1.4), and the nonzero solution $v^+(y)$ of Eqs. (1.2) with homogeneous conditions for $y = 0$ and $y = \pi$ corresponds to a flow conjugate to a uniform flow. According to formula (1.1) and condition 1, the function F from (1.2) admits a representation in the form

$$F = v_{yy} + \lambda v + \sigma\left(v_{xx} + \lambda \rho_0'(y+v)v - (y+v)v_{yy} - v_y - \frac{1}{2}v_y^2\right) + \sigma^2 F_1$$

with a smooth function F_1 of the class C^{k-1} and the coefficient $\rho_0(y) = \rho_1(y,0)$, which characterizes the fine stratification structure in the background of the basic density distribution. Since the density $\rho(y,\sigma)$ is determined only for $y \in [0,\pi]$, the values of the dimensionalized stream function ψ should lie in the same gap:

$$0 \leqslant y + v(x,y) \leqslant \pi, \qquad (x,y) \in \Omega. \tag{1.5}$$

In addition, the data at infinity should satisfy the matching conditions following from the conservation laws. The established form of the dependences of ρ and B on ψ automatically leads to the coincidence of the mass and energy fluxes for any pair of solutions of Eqs. (1.2) and (1.3) depending only on y. The situation with the law of conservation of momentum is different. The corresponding matching condition is naturally obtained by means of the variational principle [11], according to which (1.2) is the Euler–Lagrange equation for the functional with action density

$$L = -\frac{1}{2}\rho\,|\nabla_\sigma v|^2 + \sigma^{-1}\lambda \int\limits_y^{y+v} (\rho(\psi,\sigma) - \rho(y+v,\sigma))\, d\psi.$$

Owing to the invariance of this Lagrangian under the transition group relative to x, according to the Noether theorem, the differential operator F in (1.2) admits a transformation to the following divergent form:

$$v_x F(v;\sigma,\lambda) = D_x\left(L - v_x L_{v_x}\right) + D_y(-v_x L_{v_y})$$

(hereafter, the arguments Dv, D^2v, and y of the function F are omitted for brevity). With allowance for the conditions at the bottom and the lid, the integration of this equality over y shows that, for any x, the relation

$$l(v;\,\sigma,\lambda) \equiv \int\limits_0^\pi (L + \sigma\rho v_x^2)\, dy = \text{const} \tag{1.6}$$

should hold. For solutions with asymptotic behavior (1.4), as $x \to -\infty$, the constant in (1.6) is zero, and we have a restriction for the data for $x = +\infty$: all the states v^+ conjugate to the uniform flow v^- are the critical points of the functional l [considered for the functions $v = v(y)$], but only the states that lie at the same level surface $l = 0$ as the ground state are admissible.

2. Conjugate Flows. As a bifurcation problem, the problem of conjugate stratified flows was formulated by Benjamin [12], but the problem of compatibility of the three physical conservation laws has not yet been investigated. We consider the nonlinear eigenvalue problem for $v = v^+(y)$

$$(\rho v_y)_y - \rho'\left(\sigma^{-1}\lambda v + \frac{1}{2}v_y^2\right) = 0, \qquad v(0) = v(\pi) = 0 \tag{2.1}$$

with additional matching condition (1.6). For $\sigma = 0$, the unperturbed operator $F(v;0,\lambda) = v_{yy} + \lambda v$ generates a countable family of modes of the eigenfunctions and eigenvalues $v_n^+ = b\sin ny$ and $\lambda_n = n^2$ ($b \in \mathbb{R}$, $n = 1, 2, 3, \ldots$). The perturbed solution $v^+(y) = b\sin ny + bw(y)$ is found in the class of functions $C_0^k[0,\pi] = \{v \in C^k\colon v(0) = v(\pi) = 0\}$ (k from Condition 1) for Froude numbers λ close to one of the eigenvalues λ_n. Let Q_n be the orthogonal projector onto the addition to $\sin ny$ in $L_2[0,\pi]$ and $\mu(v;\,\sigma,\lambda) = \int\limits_0^\pi F(v(y);\sigma,\lambda)\,\sin ny\, dy$ be the defect functional which specifies the branching Lyapunov–Schmidt equation for problem (2.1). The function $w \in Q_n C_0^k[0,\pi]$ and the real parameters b, σ, and λ should satisfy the system

$$w_{yy} + n^2 w = Q_n f^+(w;b,\sigma,\lambda); \tag{2.2}$$

$$l(v^+;\sigma,\lambda) = 0, \qquad \mu(v^+;\sigma,\lambda) = 0, \tag{2.3}$$

where, for $b \neq 0$, the right-hand part f^+ has the form

$$bf^+(w;b,\sigma,\lambda) = F(v^+;0,n^2) - F(v^+;\sigma,\lambda)$$

and is predetermined with respect to the continuity at the point $b = 0$. We note that, for $v = v(y)$, the Lagrangian $L(v;\sigma,\lambda)$ has the structure

$$L = \frac{1}{2}(\lambda v^2 - v_y^2) + \sigma\left\{\frac{1}{2}(y+v)v_y^2 + \lambda \int\limits_y^{y+v} (\rho_0(y+v) - \rho_0(\psi))\, d\psi\right\} + \sigma^2 L_1$$

with a regular residual L_1 as $\sigma \to 0$. One can easily see that, for $\sigma = 0$ and $\lambda = \lambda_n$, both equations in (2.3) are valid with $w = 0$ and for an arbitrary real b. The restriction on the amplitude parameter b follows from condition (1.5); here we require the fulfillment of the more strict condition

$$(n\cos ny + w'(y))b > -1,$$

according to which there is no return flow of fluid in the conjugate flow. For w from the sphere $B_\delta = \{w \in$

$Q_nC_0^k[0,\pi] : \|w\|_{C^k} < \delta\}$, this restriction is obviously satisfied provided $|b| < 1/(n+\delta)$. With this in mind, in the space of the parameters we separate the domain

$$\Pi_n(\delta) = \{(b,\sigma,\lambda): \quad |b| < 1/(n+\delta), \quad \sigma > 0, \quad \sigma + |\lambda - \lambda_n| < \delta\}.$$

Lemma 1. *There is $\delta > 0$, such that, for $(b,\sigma,\lambda) \in \Pi_n(\delta)$, Eq. (2.2) has a unique solution w in the sphere B_δ. The mapping $(b,\sigma,\lambda) \to w$ is smooth and, for (σ,λ) close to $(0,n^2)$, has the asymptotic behavior*

$$w(y;b,\sigma,\lambda) = -\sigma b^{-1}\int_0^\pi \mathcal{G}_n(y,z)Q_nF_\sigma(b\sin nz;0,n^2)\,dz + O(\sigma^2 + \sigma|\lambda - n^2|),$$

where

$$\mathcal{G}_n(y,z) = \frac{1}{\pi n}(z-\pi)\sin ny\cos nz + \frac{1}{n}\begin{cases}\sin n(y-z) & (0<z<y),\\ 0 & (y<z<\pi),\end{cases}$$

and the estimate of the residual is uniform relative to b.

To prove Lemma 1, Eq. (2.2) is reduced to a nonlinear integrodifferential equation with the Green function $\mathcal{G}_n$. The existence and uniqueness of the solution for quite small δ and the uniform property of the asymptotic solution follow from the estimate

$$\|f^+(w_1;b,\sigma,\lambda) - f^+(w_2;b,\sigma,\lambda)\|_{C^{k-2}} \leqslant C(\sigma + |\lambda-\lambda_n|)\|w_1 - w_2\|_{C^k},$$

in which the constant C depends only on δ and the C^k-norm of the function ρ.

Now we consider system (2.3), in which v^+ is determined through the function w described in Lemma 1. Let l^+ and μ^+ denote the superposition of the functionals l and μ with the above mapping v^+. For the parameters b and $\boldsymbol{a} = (\sigma, \lambda - \lambda_n)$, the system has the form

$$A_n(b)\boldsymbol{a} = \boldsymbol{X}(\boldsymbol{a};b) \tag{2.4}$$

with the Jacobi matrix

$$A_n = \frac{\partial(l^+,\mu^+)}{\partial(\sigma,\lambda)}\bigg|_{\sigma=0,\lambda=\lambda_n}$$

and the smooth vector function $\boldsymbol{X}$: $\Pi_n(\delta) \to \mathbb{R}^2$, which admits the estimate $|\boldsymbol{X}(\boldsymbol{a};b)| \leqslant C|\boldsymbol{a}|^2$ uniform relative to b. Since $(w,\sin ny)_{L_2[0,\pi]} = 0$, the element w, which has the same order of smallness as σ according to Lemma 1, does not contribute to the linear part of Eq. (2.4). In view of this and owing to the potential character of the operator F (which is the gradient of the functional l), it follows that A_n has the structure of the Wronskian

$$A_n(b) = \begin{pmatrix} s_n(b) & m_n(b) \\ s_n'(b) & m_n'(b)\end{pmatrix}$$

with coefficients $m_n(b) = \pi b^2/4$ and

$$s_n(b) = n^2\int_0^\pi\int_y^{y+b\sin ny}(\rho_0(y+b\sin ny) - \rho_0(\psi))\,d\psi\,dy + \frac{1}{8}(\pi n b)^2 + \frac{n}{6}(1-(-1)^n)b^3.$$

We denote the Wronskian of the functions s_n and m_n by $\Delta_n(b) = \det A_n(b)$:

$$\Delta_n(b) = -\frac{1}{4}\pi b^4\Big(\frac{s_n(b)}{b^2}\Big)'. \tag{2.5}$$

This function plays a determining role in the subsequent considerations. If b_0 is such that $\Delta_n(b_0) \neq 0$ in a sufficiently small half-neighborhood of the point $(b_0,0,n^2) \in \partial\Pi_n(\delta)$, system (2.4) has no solutions $(b,\sigma,\lambda) \in \Pi_n(\delta)$ with $\boldsymbol{a} \neq 0$; therefore, nontrivial solutions can be found only for values of b near the zeros of the function Δ_n. The sufficient condition of the existence of solutions is given by the following statement.

Theorem 1. *Let $b_0 \in (-1/(n+\delta), 1/(n+\delta))$ be the root of the function $\Delta_n(b)$, for which the conditions*

$$\text{(i)} \quad \Delta_n'(b_0) \neq 0 \quad \textit{if} \quad b_0 \neq 0,$$

$$\text{(ii)} \quad \Delta_n^{(4)}(b_0) \neq 0 \quad \textit{if} \quad b_0 = 0$$

are satisfied. Then, for given b_0, there is a unique σ-continuous branch of conjugate states for which $(v_n^+(y;\sigma),\, \lambda_n^+(\sigma)) \to (b_0 \sin ny,\, n^2)$ in $C_0^k \times \mathbb{R}$ as $\sigma \to +0$. The dependence on σ is smooth, and the eigenvalues have the asymptotic behavior

$$\lambda_n^+(\sigma) = n^2 - \frac{s_n(b_0)}{m_n(b_0)}\sigma + O(\sigma^2). \tag{2.6}$$

Proof. For $b_0 \neq 0$, the matrix $A_n(b_0)$ has a one-dimensional null space and a null cospace generated by the vectors $\boldsymbol{e} = (m_n(b_0), -s_n(b_0))$ and $\boldsymbol{e}_* = (m_n'(b_0), -m_n(b_0))$. Hence, system (2.4) is equivalent to one branching equation for b and $\tau = |\boldsymbol{e}|^{-2}\boldsymbol{a}\cdot\boldsymbol{e}$, which takes the form $t_1(b-b_0) + t_2\tau + \Gamma(b,\tau) = 0$ after the trivial solution $\tau = 0$ branches. Here $t_1 = A_n'(b_0)\boldsymbol{e}\cdot\boldsymbol{e}_*$, the form of the coefficient t_2 is not important, and $\Gamma(b,\tau) = O(\tau^2 + (b-b_0)^2)$. It follows from the expressions for $\boldsymbol{e}$ and $\boldsymbol{e}_*$ that $t_1 = m_n(b_0)\Delta_n'(b_0)$; consequently, in the case of a simple nonzero root of the function Δ_n there is a unique nontrivial branch of solutions $b(\tau)$. Since $\boldsymbol{a} = \tau\boldsymbol{e} + O(\tau^2)$ and the first component $\boldsymbol{e}$ is different from zero, one can use σ as the free parameter instead of τ, which simultaneously yields the asymptotic solution (2.6).

Now we consider the root $b_0 = 0$ of the Wronskian Δ_n, which is at least four-multiple; since $s_n(b) = O(b^2)$ as $b \to 0$, the matrix $A_n(0)$ is zero. The ambiguity is easily eliminated if one passes from (2.4) to an equivalent system of the same form with the matrix

$$B_n(b) = \begin{pmatrix} b^{-2}s_n(b) & b^{-2}m_n(b) \\ b^{-1}s_n'(b) & b^{-1}m_n'(b) \end{pmatrix}$$

instead of A_n. The smooth vector function $\boldsymbol{X}$, which appears after this transformation on the right-hand side, still has, by virtue of Lemma 1, the uniform asymptotic estimate relative to b as $|\boldsymbol{a}| \to 0$. Because $\det B_n(0) = 0$ and $\operatorname{rang} B_n(0) = 1$, with allowance for the above remarks the further analysis is similar to that in condition (i). Condition (ii) is equivalent to the inequality $t_1 \neq 0$ in the branching equation, and formula (2.6) holds in this case as well if ones passes to the limit $b_0 \to 0$ in the coefficient at σ. Theorem 1 is proved.

Remark. If the multiplicity of the root b_0 is such that condition (i) or (ii) is violated, the number of branches of conjugate flows born in each of the eigenvalues, and, as $\sigma \to 0$, their asymptotic behavior can be determined by means of the Newton diagram with the use of the nonzero coefficients at higher degrees τ and $b - b_0$ in the branching equation.

3. Spectrum of the Linear Problem. We clarify the mutual arrangement of the branches of conjugate states in the plane (σ, λ) and of the spectrum of the linear problem of small perturbations of the ground state for $x = -\infty$. Equation (1.2), linearized on the trivial solution has the form

$$\operatorname{div}_\sigma(\rho\nabla_\sigma v) - \lambda\sigma^{-1}\rho' v = f, \tag{3.1}$$

where $\rho = \rho(y,\sigma)$ is the defined function from (1.1) and $\rho' = \rho_y$. The homogeneous equation has solutions of the plane-wave type

$$v_n(x,y;\,\sigma,\text{æ}) = e^{i\text{æ}x}\,\varphi_n(y;\,\sigma,\text{æ}), \qquad \text{æ} \in \mathbb{R},\ n \in \mathrm{N}, \tag{3.2}$$

where φ_n are eigenfunctions of the Sturm–Liouville problem

$$(\rho\varphi_y)_y - (\sigma\text{æ}^2\rho + \lambda\sigma^{-1}\rho')\varphi = 0, \qquad \varphi(0) = \varphi(\pi) = 0. \tag{3.3}$$

If Condition 1 is satisfied, all its eigenvalues λ_n are real and positive. It is known [10] that $\lambda_n(\sigma,\text{æ})$ monotonically increase with æ^2; note that $\lambda_n \to +\infty$ as $\text{æ}^2 \to +\infty$. It is clear that solutions of the form (3.2) are possible only for (σ,λ) belonging to the set

$$\Sigma = \{(\sigma,\lambda) \mid \sigma \in (0,\sigma_0],\, \lambda \geqslant \lambda_1(\sigma,0)\},$$

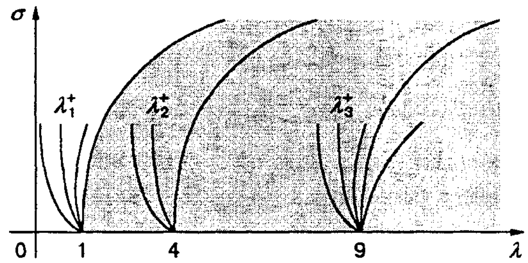

Fig. 2

where λ_1 is the smallest eigenvalue. If $\lambda < \lambda_1(\sigma, 0)$, Eq. (3.1) with $f \in L_2(\Omega)$ is unambiguously resolved in the space $W^2_{2,0}(\Omega) = \{v \in W^2_2(\Omega) | \, v(x,0) = v(x,\pi) = 0\}$. This can be established by applying the Fourier transform with respect to x to (3.1) and using the expansion $\hat{v}(\xi, y)$ into the eigenfunctions $\varphi_n(y;\, \sigma, \xi)$, which form the orthogonal basis in $L_2[0,\pi]$ with respect to the scalar product with weight $-\sigma^{-1}\rho_y(y,\sigma)$ and the basis in $\overset{\circ}{W}{}^1_2\,[0,\pi]$ orthogonal in the scalar product

$$[u(y), v(y)]_\xi = \int\limits_0^\pi \rho(y,\sigma)(u'(y)v'(y) + \sigma\xi^2 u(y)v(y))\, dy.$$

According to Sturm's comparison theorem, the eigenvalues admit the estimate from below

$$\lambda_n(\sigma,\xi) \geqslant \frac{r(\sigma)}{R(\sigma)}\,(n^2 + \sigma\xi^2)$$

with quantities $r = \min\limits_{y\in[0,\pi]} \rho(y,\sigma)$ and $R = \max\limits_{y\in[0,\pi]}(-\sigma^{-1}\rho_y(y,\sigma))$; therefore, for v, the estimate $\|v\|_{W^2_2} \leqslant C(\sigma,\lambda)\|f\|_{L_2}$, which holds for the points (σ,λ) outside Σ, holds. It follows from the aforesaid that, for each fixed $\sigma \in (0,\sigma_0]$, the symmetric operator in (3.1) with the domain of definition $W^2_{2,0}(\Omega)$ has a continuous spectrum filling the real semi-axis $\operatorname{Re}\lambda \geqslant \lambda_1(\sigma,0)$ in the plane of complex λ. The set Σ unites these spectra in the plane of real pairs (σ,λ) relative to the parameter σ; it is arranged in such a way that during each transition through the smooth curve $\Lambda_n : \lambda = \lambda_n(\sigma,0)$ toward the increase in λ, the available generalized eigenfunctions $v_m(x,y;\, \sigma, \pm|æ|)$ $(m = 1,2,\ldots,n-1)$ of the form (3.2) are added by a pair of functions of the mode with number n. The pattern of spectrum arrangement is well Illustrated by the case of exponential stratification $\rho = \exp(-\sigma y)$, for which $\varphi_n(y;\, \sigma, æ) = \sqrt{2/\pi}\, e^{\sigma y/2} \sin ny$ and $\lambda_n(\sigma, æ) = n^2 + \sigma æ^2 + \sigma^2/4$, so that each of the curves Λ_n is the parabola $\lambda = n^2 + \sigma^2/4$.

In the long-wave limit ($æ \to 0$), the Sturm–Liouville problem (3.3) coincides with the equations of conjugate flows (2.1) linearized on the zero solution. As a consequence, the curve Λ_n emerges from the point $(0,n^2)$ on the λ axis at which a fan of branches of the nth mode of conjugate flows grows according to Theorem 1. Calculating the perturbations of the eigenvalue $\lambda_n(\sigma,0)$ with respect to the small parameter σ, for the slope of the curve Λ_n at the bifurcation point we have the expression

$$D_\sigma \lambda_n(0,0) = -\frac{2n^2}{\pi}\int\limits_0^\pi \rho_0'(y)\sin ny\, dy - \frac{\pi n^2}{2},$$

which coincides with the quantity $-s_n''(0)/m_n''(0)$. A comparison with the asymptotic solution (2.6) shows that at the bifurcation point, the curve Λ_n touches the branch of conjugate states which corresponds to the root $b_0 = 0$ of the Wronskian $\Delta_n(b)$. The curves $(\sigma, \lambda_n^+(\sigma))$ generated by the nonzero roots b_0 branch outside the set $\Sigma_n = \{(\sigma,\lambda)\colon \lambda \geqslant \lambda_n(\sigma,0)\}$ if the inequality

$$\frac{s_n(b_0)}{m_n(b_0)} > \frac{s_n''(0)}{m_n''(0)} \tag{3.4}$$

holds, and inside Σ_n if the opposite strict inequality holds (the spectrum and the branches of conjugate flows are shown in Fig. 2).

4. The Bore Structure. We consider in more detail the first mode of conjugate states: only for it can the branches of conjugate flows located outside the spectrum Σ of the linearized problem exist. We fix one of the simple nonzero roots of the Wronskian Δ_1; according to Theorem 1, the branch of shear flows $[\sigma, \lambda_1^+(\sigma)]$ corresponds to it. In Eq. (1.2), we set $\lambda = \lambda_1^+(\sigma)$ and take the function $v^+(y;\sigma)$ as a limiting function in (1.4). Below, we construct an approximate solution of problem (1.2)–(1.4) with the specified behavior at infinity. It is easy to establish the form of the principal term of the asymptotic behavior of the desired solution $v(x,y;\sigma)$ as $\sigma \to 0$ assuming that v is smooth with respect to σ up to the value of $\sigma = 0$. The functions $v_0 = v(x,y;\,0)$ and $v_1 = D_\sigma v(x,y;\,0)$ should satisfy the equations

$$D_y^2 v_j + v_j = f_j \quad (x \in \mathbb{R},\ 0 < y < \pi), \qquad v_j = 0 \quad (y = 0,\ y = \pi),$$

where $f_0 = 0$, $f_1 = -F_\sigma(v_0;\,0,\,1) - D_\sigma \lambda_1^+(0)\,F_\lambda(v_0;\,0,\,1)$. In the zero approximation, we have $v_0 = a_0(x)\sin y$, where the function a_0 is determined from the resolvability condition of the inhomogeneous problem for v_1: the right-hand part of f_1 for each $x \in \mathbb{R}$ should be orthogonal to $\sin y$ in $L_2([0,\pi])$. It gives the equation

$$a_0'' + p'(a_0) = 0 \tag{4.1}$$

with function p; by virtue of formula (2.6) for λ_1^+, the potentiality property of the operator F, and the determination of the coefficients s_1 and m_1, this function has the form

$$p(b) = \frac{2}{\pi}\Big(s_1(b) - \frac{s_1(b_0)}{m_1(b_0)}\,m_1(b)\Big).$$

According to (2.5), this function is expressed through the Wronskian $\Delta_1(b)$ as follows:

$$p(b) = \frac{8}{\pi^2}\,b^2 \int\limits_b^{b_0} t^{-4}\Delta_1(t)\,dt. \tag{4.2}$$

Equation (4.1) can have limited solutions which damp as $x \to -\infty$ only for sign-definite functions p and, therefore, we make the following assumption.

Condition 2. The simple root b_0 of the functions Δ_1 is such that the inequality

$$\frac{s_1(b_0)}{m_1(b_0)} > \frac{s_1(b)}{m_1(b)}$$

holds everywhere in the interval between $b = 0$ (including this point) and $b = b_0$. This requirement imposes restrictions only on the coefficient $\rho_0 = \rho_1(y,0)$ in formula (1.1), by which the function Δ_1 is completely determined. This restriction is satisfied if, for example, b_0 is the nearest root to the point $b = 0$, and the sign of $\Delta_1(b)$ is opposite to that of b_0. According to Condition 2, inside the gap considered we have $p(b) < 0$, and its ends $b = 0$ and $b = b_0$ are exactly the two-multiple roots $p(b)$. Under the adopted assumptions, the desired solution a_0 is given by the quadrature

$$x = \operatorname{sign} b_0 \int\limits_{b_*}^{a_0} \frac{db}{\sqrt{-2p(b)}},$$

where $b_* = a_0(0) \in (0, b_0)$ is fixed by the choice of the reference system. The function $a_0(x)$ is strictly monotone, takes on the values from 0 to b_0 as x varies from $-\infty$ to $+\infty$, and has the exponential behavior

$$|a_0(x)| \leqslant C\exp(-\alpha_0|x|), \qquad |b_0 - a_0(x)| \leqslant C\exp(-\beta_0|x|)$$

with indices $\alpha_0 > 0$ and $\beta_0 > 0$,

$$\alpha_0^2 = -\frac{16}{\pi^2}\int\limits_0^{b_0} t^{-4}\Delta_1(t)\,dt, \qquad \beta_0^2 = \frac{8}{\pi^2 b_0^2}\,\Delta_1'(b_0).$$

Condition 2 implies satisfaction of the inequality (3.4); therefore, the approximate solution obtained describes a continuous bore spreading over the homogeneous state on the left at infinity with supercritical speed whose square is equal to $c^2 = \sigma gh/(\pi - 4\sigma b_0^{-2} s_1(b_0))$ with an accuracy of the order $O(\sigma^3)$. For conjugate states generated by the modes with $n \geqslant 2$, the bifurcation curves $(\lambda_n^+(\sigma), \sigma)$ are inside the spectrum of linear waves. Here, most probably, the bore should be present coupled with a periodic wave whose profile is determined by the generalized eigenfunction (3.2) in the principal term of the asymptotic solution. This situation is similar to that occurring in the surface-wave problem with account of the capillarity for Bond numbers smaller than one third [13], for which the existence of stationary configurations in the form of solitary waves with oscillating tails at infinity is strictly proved.

5. Examples. We consider the density profiles (1.1), for which the existence conditions of conjugate flows and bore-type waves are satisfied. Most readily, the presence of simple nonzero roots of the Wronskian Δ_1 can be established for polynomial dependences of the coefficient $\rho_0(y)$. If the degree $\rho_0(y)$ is not greater than two, Δ_1 has no roots different from zero. It follows from Theorem 1 that, for a uniform flow with a purely exponential or linear density distribution, there can be only close conjugate states with amplitude parameter $b(\sigma) \to 0$ as $\sigma \to 0$. The case of higher-order polynomials is more interesting:

$$\rho_0(y) = \sum_{k=1}^{n} r_{k-1} y^k \qquad (n > 2).$$

In this case, Δ_1 has the form

$$\Delta_1(b) = -\frac{1}{12}\pi b^4 \sum_{k=0}^{n-2} d_{k+1} b^k, \tag{5.1}$$

where the vector of the coefficients $\boldsymbol{d} = (d_1, \ldots, d_{n-1})$ is linearly expressed via $\boldsymbol{r} = (r_1, \ldots, r_{n-1})$ (the omitted coefficient r_0 does not affect the form of Δ_1) by the formula $\boldsymbol{d} = T\boldsymbol{r} + \boldsymbol{s}$ with the vector $\boldsymbol{s} = (1, 0, \ldots, 0)$ and the upper triangular matrix T whose coefficients are strictly positive on the principal diagonal and above it:

$$t_{ks} = 3C_s^k \frac{k(s+1)}{k+2} \int_0^{\pi} y^{s-k} \sin^{k+2} y\, dy \qquad (s \geqslant k).$$

Since the transformation T for any polynomial of the form (5.1) is invertible, one can always indicate a stratification law (1.1) according to which the function Δ_1 is a determinant of the matrix of system (2.4). Let $n = 2m + 3$ with integer nonnegative m, and all the nonzero roots of the Wronskian form a geometrical progression $b_j = q^{2m+2-j}$ $(j = 1, \ldots, 2m+1)$. For this Δ_1 and each pair of the neighboring intervals (b_{2i-1}, b_{2i}) and (b_{2i}, b_{2i+1}) $(i = 1, \ldots, m)$, the equality

$$\int_{b_{2i-1}}^{b_{2i}} t^{-4}\Delta_1(t)\, dt = -\int_{b_{2i}}^{b_{2i+1}} t^{-4}\Delta_1(t)\varphi_m(t)\, dt$$

with the function $\varphi_m(t) = q^{2m+2}(1-t)/(1-q^{2m+1})$ holds. If q is chosen in the limit $0 < q < 1/2$, the inequalities $\Delta_1(t) < 0$ and $0 < \varphi_m(t) < 1$ will be satisfied simultaneously on any of the intervals (b_{2i}, b_{2i+1}) Hence, in formula (4.2), one can use any of the roots b_j with odd number $j = 2i + 1$ from $m + 1$ as b_0. This example shows that a small perturbation of the linear or exponential stratification can lead to the appearance of any already specified number of branches of conjugate states of the first mode with finite amplitude $b(\sigma)$ which does not vanish in the limit $\sigma \to 0$. Here, each branch is conjugated to the basic uniform flow by its stationary bore. The strong sensitivity of nonlinear wave structures to small perturbations of stratification was noted in [7, 14].

In the particular case $m = 0$, the determinant $\Delta_1(b)$ is the fifth-order polynomial, and the single nonzero real root

$$b_0 = -\frac{16}{27}\Big(4 + \frac{8r_1 + 3}{3\pi r_2}\Big)$$

is possible for it. The conjugate flow and the approximate solution in the form of a bore exist for values of the coefficients r_1 and $r_2 > 0$ for which the inequalities $|b_0| < 1$ and $b_0 \neq 0$ are satisfied. In this case,

$$p(b) = -\frac{9}{16}\pi r_2 b^2 (b_0 - b)^2,$$

so that Eq. (4.1) is the first integral of the modified Korteweg–de Vries equation with cubic nonlinearity, and the wave profile in the zero approximation has the form

$$a_0(x) = \frac{b_0}{2}\Big(1 + \tanh\frac{3}{8}\sqrt{\pi r_2}\, b_0 x\Big).$$

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REFERENCES

1. L. V. Ovsyannikov, N. I. Makarenko, V. I. Nalimov, et al., *Nonlinear Problems of the Theory of Surface and Internal Waves* [in Russian], Nauka, Novosibirsk (1985).
2. M. Funakoshi, "Long internal waves in a two-layer fluid," *J. Phys. Soc. Jpn.*, **54**, No. 1, 128–144 (1985).
3. C. J. Amick and R. E. L. Turner, "Small internal waves in two-fluid systems," *Arch. Rational Mech. Anal.*, **108**, No. 2, 111–139 (1989).
4. N. I. Makarenko, "Smooth bore in a two-layer fluid," *Int. Ser. Numer. Math.*, **106**, 195–204 (1992).
5. A. Mielke, "Homoclinic and heteroclinic solutions in two-phase flow," *Adv. Ser. Nonlinear Dyn.*, Vol. 7: Proc. IUTAM/ISIMM Symp. on Structure and Dynamics of Nonlinear Waves in Fluids, World Scientific (1995), pp. 353–362.
6. J. W. Miles, "On internal solitary waves," *Tellus*, **31**, No. 5, 456–462 (1979).
7. A. A. Borisov and O. G. Derzho, "The structure of steady-state solitary waves of finite amplitude," *Izv. Akad. Nauk SSSR, Ser. Tekh. Nauk.*, No. 2, 60–70 (1990).
8. N. V. Gavrilov, "Internal solitary waves immovable in a laboratory coordinate system and smooth bores," *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 1, 29–33 (1994).
9. V. I. Bukreev and A. V. Gusev,"Forced smooth bore in a continuously stratified fluid," *Dokl. Ross. Akad. Nauk*, **363**, No. 3, 327–329 (1998).
10. Yih Chia-Shun, *Stratified Flows*, Academic Press, New York (1980).
11. T. B. Benjamin, "Impulse, flow force and variational principles," *IMA J. Appl. Math.*, **32**, 3–68 (1984).
12. T. B. Benjamin, "A unified theory of conjugate flows," *Philos. Trans. R. Soc. London, Ser. A*, **269**, 587–643 (1971).
13. J. T. Beale, "Exact solitary water waves with capillary ripples at infinity," *Commun. Pure Appl. Math.*, **44**, 211–257 (1991).
14. D. J. Benney and D. R. S. Ko, "The propagation of long large amplitude internal waves," *Stud. Appl. Math.*, **59**, 187–199 (1978).